Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Time 1 hour 40 minutes

Paper reference **8FM0/01**

Further Mathematics

Advanced Subsidiary
PAPER 1: Core Pure Mathematics

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 80.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 7 & 2 \\ -5 & 8 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 6 & 5 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{pmatrix}$$

Given that $\mathbf{I}$ is the 3×3 identity matrix,

(a) (i) show that there is an integer k for which

$$\mathbf{AB} - 3\mathbf{C} + k\mathbf{I} = \mathbf{0}$$

stating the value of k

(ii) explain why there can be no constant m such that

$$\mathbf{BA} - 3\mathbf{C} + m\mathbf{I} = \mathbf{0}$$

(4)

(b) (i) Show how the matrix $\mathbf{C}$ can be used to solve the simultaneous equations

$$\begin{aligned} -5x + 2y + z &= -14 \\ 4x + 3y + 8z &= 3 \\ -6x + 11y + 2z &= 7 \end{aligned}$$

(ii) Hence use your calculator to solve these equations.

(3)

1. a) (i) $AB - 3C + KI = 0$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} ag+bj & ah+bk & ai+bl \\ cg+dj & ch+dk & ci+dl \\ eg+fj & eh+fk & ei+fl \end{bmatrix}$$

$[3 \times 2]$ $\quad [2 \times 3]$ $\quad [3 \times 3]$

rows columns

$$AB = \begin{bmatrix} 4 & -1 \\ 7 & 2 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ -1 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 8+1 & 12-6 & 8-5 \\ 14-2 & 21+12 & 14+10 \\ -10-8 & -15+48 & -10+40 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 3 \\ 12 & 33 & 24 \\ -18 & 33 & 30 \end{bmatrix}$$

Question 1 continued

$$AB - 3C = \begin{bmatrix} 9 & 6 & 3 \\ 12 & 33 & 24 \\ -18 & 33 & 30 \end{bmatrix} - 3\begin{bmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 3 \\ 12 & 33 & 24 \\ -18 & 33 & 30 \end{bmatrix} - \begin{bmatrix} -15 & 6 & 3 \\ 12 & 9 & 24 \\ -18 & 33 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

3×3 identity

$$AB - 3C + KI = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} + K\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} + \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix} = 0$$

$\therefore K = -24$

(ii)

$$\underset{[2\times 3]}{\begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix}} \times \underset{[3\times 2]}{\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}} \rightarrow \text{gives you } [2\times 2] \text{ matrix}$$

BA

rows columns

$BA - 3C$ would be a $[2\times 2]$ matrix $-$ $[3\times 3]$ matrix

Because they have different dimensions, you cannot subtract 3C from BA

Question 1 continued

b)(i) $-5x + 2y + z \qquad -14$
$4x + 3y + 8z = 3$
$-6x + 11y + 2z \qquad 7$

$$\begin{bmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -14 \\ 3 \\ 7 \end{bmatrix}$$

$$C\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -14 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1}\begin{bmatrix} -14 \\ 3 \\ 7 \end{bmatrix}$$

inverse matrix C^{-1}

(ii) $C^{-1} = \frac{1}{360}\begin{bmatrix} -82 & 7 & 13 \\ -56 & -4 & 44 \\ 62 & 43 & -23 \end{bmatrix}$ (by calculator)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{360}\begin{bmatrix} -82 & 7 & 13 \\ -56 & -4 & 44 \\ 62 & 43 & -23 \end{bmatrix}\begin{bmatrix} -14 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3 \\ -5/2 \end{bmatrix} \qquad x = \frac{7}{2},\ y = 3,\ z = -\frac{5}{2}$$

2. (a) Express the complex number $w = 4\sqrt{3} - 4\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$ where $r > 0$ and $-\pi < \theta \leqslant \pi$

(4)

(b) Show, on a single Argand diagram,

(i) the point representing w

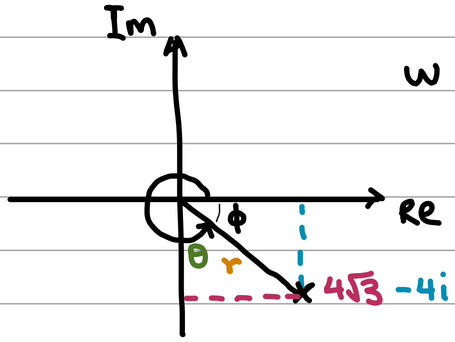
(ii) the locus of points defined by $\arg(z + 10\text{i}) = \dfrac{\pi}{3}$

(3)

(c) Hence determine the minimum distance of w from the locus $\arg(z + 10\text{i}) = \dfrac{\pi}{3}$

(3)

a)



argument

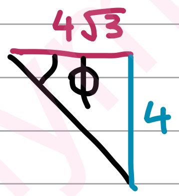
$w = r(\cos\theta + i\sin\theta)$ (angle formed anticlockwise from the positive real axis)

modulus

$r = |w|$ (modulus of $x+iy = \sqrt{x^2+y^2}$)

$= \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{48+16} = \sqrt{64} = 8$

$\theta = 2\pi - \phi$



$\tan\phi = \dfrac{\text{opp.}}{\text{adj.}} = \dfrac{\cancel{4}}{\cancel{4}\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\theta = 2\pi - \dfrac{\pi}{6}$

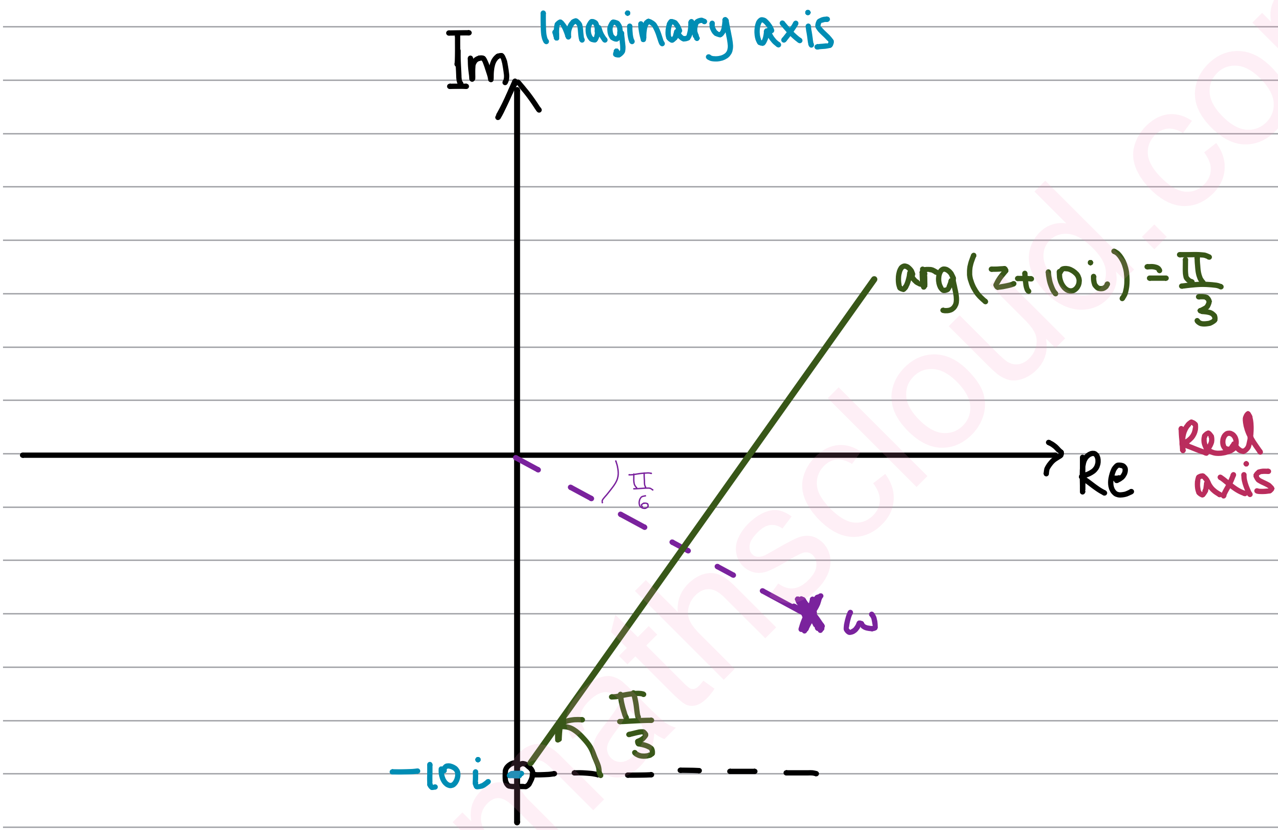
$\therefore \phi = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

$\theta = \dfrac{11\pi}{6}$ or $-\dfrac{\pi}{6}$

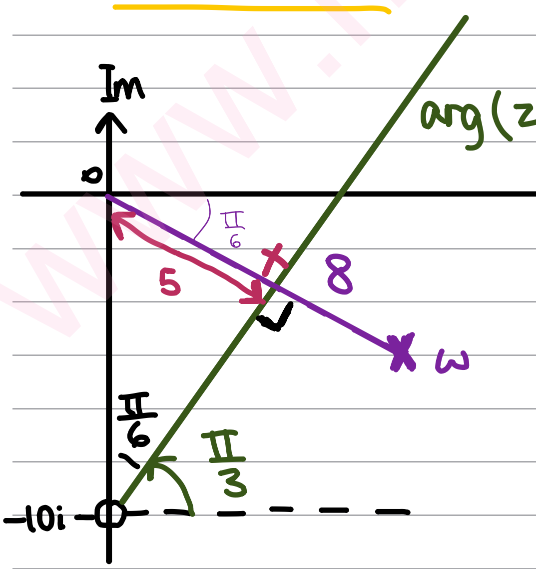
$w = 8\left(\cos\left(-\dfrac{\pi}{6}\right) + i\sin\left(-\dfrac{\pi}{6}\right)\right)$

b) (i) coordinate of w is $(4\sqrt{3}, -4)$

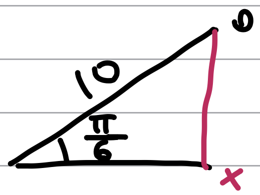
(ii) $\arg(z+10i) = \frac{\pi}{3} = \arg(z-(-10i))$



c) METHOD 1 (using trig. to find distance)



• Line from origin is perpendicular to the locus
∴ shortest distance $= WX$



$OX = 10\sin\frac{\pi}{6} = 10 \times \frac{1}{2} = 5$

$\therefore WX = 8 - 5 = 3$ ← minimum distance

Question 2 continued

METHOD 2 (finding points of intersection (x))

equation of locus : gradient $= \frac{\Delta y}{\Delta x} = \frac{\text{opp.}}{\text{adj.}} = \tan\frac{\pi}{3} = \sqrt{3}$

$\hookrightarrow y - y_1 = m(x - x_1)$

using point $(0, -10)$

$$y - (-10) = \sqrt{3}(x - 0)$$

$$y + 10 = \sqrt{3}x$$

$$y = \sqrt{3}x - 10$$

equation of OW gradient $= \frac{\Delta y}{\Delta x} = \frac{\text{opp.}}{\text{adj.}} = \tan -\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$

$\hookrightarrow y - y_1 = m(x - x_1)$

$$y - 0 = -\frac{\sqrt{3}}{3}(x - 0)$$

$$y = -\frac{\sqrt{3}}{3}x$$

X is at the intersection of the 2 lines (x_x, y_x)

$$-\frac{\sqrt{3}}{3}x_x = \sqrt{3}x_x - 10$$

$$\frac{4\sqrt{3}}{3}x_x = 10$$

$$x_x = \frac{5\sqrt{3}}{2} \qquad y_x = -\frac{\sqrt{3}}{3}\left(\frac{5\sqrt{3}}{2}\right) = -\frac{5}{2}$$

$X : \left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$

Minimum distance from W to X is :

$$\sqrt{\left(\frac{5\sqrt{3}}{2} - 4\sqrt{3}\right)^2 + \left(-\frac{5}{2} - (-4)\right)^2} = \sqrt{\frac{27}{4} + \frac{9}{4}} = 3$$

3. *With respect to the* ***right-hand rule****, a rotation through* $\theta°$ *anticlockwise about the y-axis is represented by the matrix*

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The point P has coordinates (8, 3, 2)

The point Q is the image of P under the transformation reflection in the plane $y = 0$

(a) Write down the coordinates of Q

(1)

The point R is the image of P under the transformation rotation through 120° anticlockwise about the y-axis, with respect to the **right-hand rule**.

(b) Determine the exact coordinates of R

(2)

(c) Hence find $|\overrightarrow{PR}|$ giving your answer as a simplified surd.

(2)

(d) Show that $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ are perpendicular.

(1)

(e) Hence determine the exact area of triangle PQR, giving your answer as a surd in simplest form.

(2)

a) Reflection in the plane $y=0$

means that the y co-ordinate would be reflected

$Q: (8, -3, 2)$

b) $R = [120° \text{ rotation}]\, P$

$$R = \begin{bmatrix} \cos 120 & 0 & \sin 120 \\ 0 & 1 & 0 \\ -\sin 120 & 0 & \cos 120 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4+\sqrt{3} \\ 3 \\ -4\sqrt{3}-1 \end{bmatrix}$$

$R: (-4+\sqrt{3}, 3, -4\sqrt{3}-1)$

Question 3 continued

c) $|\overrightarrow{PR}|$ is the distance between P and R

└ the distance between 2 points is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$$\sqrt{((-4+\sqrt{3})-(8))^2+((3)-(3))^2+((-4\sqrt{3}-1)-(2))^2}$$

$$=\sqrt{204} = 2\sqrt{51}$$

d) 2 vectors are perpendicular if dot product is 0

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = a_1a_2 + b_1b_2 + c_1c_2$$

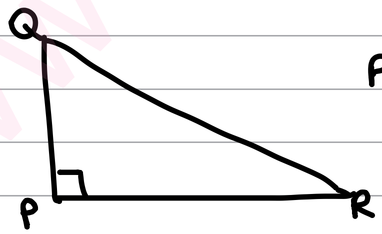
$$PQ = OQ - OP = \begin{pmatrix} 8 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix}$$

$$PR = OR - OP = \begin{pmatrix} -4+\sqrt{3} \\ 3 \\ -4\sqrt{3}-1 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -12+\sqrt{3} \\ 0 \\ 4\sqrt{3}-3 \end{pmatrix}$$

$$PQ \cdot PR = \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -12+\sqrt{3} \\ 0 \\ 4\sqrt{3}-3 \end{pmatrix} = 0$$

$\therefore$ PQ and PR are perpendicular

e) Because PQ and PR are perpendicular,



PQR is a right angle triangle

$\text{Area}_{\Delta} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}|\overrightarrow{PQ}||\overrightarrow{PR}|$

$$\text{Area} = \frac{1}{2} \times 6 \times 2\sqrt{51} = 6\sqrt{51}$$

4. The roots of the quartic equation

$$3x^4 + 5x^3 - 7x + 6 = 0$$

are α, β, γ and δ

Making your method clear and without solving the equation, determine the exact value of

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(3)

(ii) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} + \frac{2}{\delta}$

(3)

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)(3 - \delta)$

(3)

$ax^4 + bx^3 + cx^2 + dx + e = 0$

sum of roots   sum of all possible products of 2 roots $\alpha\beta\gamma\delta$

$\Sigma\alpha_i = -\frac{b}{a}$ $\Sigma\alpha_i\alpha_j = \frac{c}{a}$ $\Sigma\alpha_i\alpha_j\alpha_k = -\frac{d}{a}$ $\Sigma\alpha_i\alpha_j\alpha_k\alpha_l = \frac{e}{a}$

sum of all possible products of 3 roots

(i) $\Sigma\alpha_i = -\frac{5}{3}$ $\Sigma\alpha_i\alpha_j = \frac{0}{3}$ $\Sigma\alpha_i\alpha_j\alpha_k = \frac{7}{3}$ $\Sigma\alpha_i\alpha_j\alpha_k\alpha_l = \frac{6}{3} = 2$

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = ?$

$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$
$+ 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + 2\gamma\delta$

$(\Sigma\alpha_i)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\Sigma\alpha_i\alpha_j)$

$\left(-\frac{5}{3}\right)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(0)$

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{25}{9}$

Question 4 continued

METHOD 1 (expand brackets)

(ii) $\frac{2}{\alpha}+\frac{2}{\beta}+\frac{2}{\gamma}+\frac{2}{\delta} = ?$

$2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) = 2\left(\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}\right)$

$= 2\left(\frac{\sum\alpha_i\alpha_j\alpha_k}{\sum\alpha_i\alpha_j\alpha_k\alpha_l}\right) = 2\left(\frac{\frac{7}{3}}{2}\right) = \frac{7}{3}$

(iii) $(3-\alpha)(3-\beta)(3-\gamma)(3-\delta) = ?$

$(9-3\alpha-3\beta+\alpha\beta)(3-\gamma)(3-\delta)$

$= (9-3\alpha-3\beta+\alpha\beta)(9-3\gamma-3\delta+\gamma\delta)$

$= 81-27\gamma-27\delta+9\gamma\delta-27\alpha+9\alpha\gamma+9\alpha\delta-3\alpha\gamma\delta$
$-27\beta+9\beta\gamma+9\beta\delta-3\beta\gamma\delta+9\alpha\beta-3\alpha\beta\gamma-3\alpha\beta\delta+\alpha\beta\gamma\delta$

$= 81-27(\sum\alpha_i)+9(\sum\alpha_i\alpha_j)-3(\alpha_i\alpha_j\alpha_k)+\sum\alpha_i\alpha_j\alpha_k\alpha_l$

$= 81-27\left(-\frac{5}{3}\right)+9(0)-3\left(\frac{7}{3}\right)+2$

$= 121$

METHOD 2 (form new equation)

$u = 3-x \qquad 3x^4+5x^3-7x+6=0$
$x = 3-u$

$3(3-u)^4+5(3-u)^3-7(3-u)+6=0$

$3(81-108u+54u^2-12u^3+u^4)+5(27-27u+9u^2-u^3)-21+7u+6$

$3u^4-41u^3+207u^2-452u+363=0$

$(3-\alpha)(3-\beta)(3-\gamma)(3-\delta) = \sum u_iu_ju_ku_l = \frac{363}{3} = 121$

5. (a) Use the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n}(3r^2 - 17r - 25) = n(n^2 - An - B)$$

where A and B are integers to be determined.

(4)

(b) Explain why, for $k \in \mathbb{N}$,

$$\sum_{r=1}^{3k} r\tan(60r)^\circ = -k\sqrt{3}$$

(2)

Using the results from part (a) and part (b) and showing all your working,

(c) determine any value of n that satisfies

$$\sum_{r=5}^{n}(3r^2 - 17r - 25) = 15\left[\sum_{r=6}^{3n} r\tan(60r)^\circ\right]^2$$

(6)

a) $\sum_{r=1}^{n}(3r^2-17r-25) = 3\sum_{r=1}^{n} r^2 - 17\sum_{r=1}^{n} r - 25n$

$= 3\left(\frac{n}{6}(n+1)(2n+1)\right) - 17\left(\frac{n}{2}(n+1)\right) - 25n$

$= \frac{n}{2}(n+1)(2n+1) - \frac{17n}{2}(n+1) - 25n$

$= \frac{n}{2}(2n^2+3n+1-17n-17-50)$

$= \frac{n}{2}(2n^2-14n-66) = n(n^2-7n-33)$

$A = 7 \qquad B = 33$

b) $\sum_{r=1}^{3K} r\tan(60r)^\circ$

$= \tan 60^\circ + 2\tan 120^\circ + 3\tan 180 + 4\tan 240 + 5\tan 300 + 6\tan 360 + \ldots 3K\tan(180K)^\circ$

$= (\sqrt{3} - 2\sqrt{3} + 0) + (4\sqrt{3} - 5\sqrt{3} + 0) + \ldots 3K\tan(180K)^\circ$

$-\sqrt{3}$ $\qquad -\sqrt{3}$

Question 5 continued

Because tan has a period of 180°, $\tan(60r)°$ repeats every 3 terms

└ each group of 3 results in $-\sqrt{3}$

so from $r=1$ to $r=3K$, there are K groups of 3 terms, so the sum would be $\underline{\underline{-K\sqrt{3}}}$.

c) $\sum_{r=5}^{n}(3r^2-17r-25) = \sum_{r=1}^{n}(3r^2-17r-25) - \sum_{r=1}^{4}(3r^2-17r-25)$

$= n(n^2-7n-33) - (-180)$

$= n^3-7n^2-33n+180$

$\sum_{r=6}^{3n} r\tan(60r)° = \sum_{r=1}^{3n} r\tan(60r)° - \sum_{r=1}^{5}\tan(60r)°$

$= -n\sqrt{3} - (-2\sqrt{3})$

$= -n\sqrt{3} + 2\sqrt{3}$

$n^3-7n^2-33n+180 = 15(2\sqrt{3}-n\sqrt{3})^2$

$n^3-7n^2-33n+180 = 15(12-12n+3n^2)$

$n^3-52n^2+147n = 0$

$n(n^2-52n+147) = 0$

$n(n-49)(n-3) = 0$

$n = 0 \cup 49 \cup 3$

However n must be at least >5 for $\sum_{r=5}^{n}$ to be valid,

$\therefore \ n = 49$

6. The surface of a horizontal tennis court is modelled as part of a horizontal plane, with the origin on the ground at the centre of the court, and

- **i** and **j** are unit vectors directed across the width and length of the court respectively
- **k** is a unit vector directed vertically upwards
- units are metres

After being hit, a tennis ball, modelled as a particle, moves along the path with equation

$$\mathbf{r} = (-4.1 + 9\lambda - 2.3\lambda^2)\mathbf{i} + (-10.25 + 15\lambda)\mathbf{j} + (0.84 + 0.8\lambda - \lambda^2)\mathbf{k}$$

where λ is a scalar parameter with $\lambda \geqslant 0$

Assuming that the tennis ball continues on this path until it hits the ground,

(a) find the value of λ at the point where the ball hits the ground.

(2)

The direction in which the tennis ball is moving at a general point on its path is given by

$$(9 - 4.6\lambda)\mathbf{i} + 15\mathbf{j} + (0.8 - 2\lambda)\mathbf{k}$$

(b) Write down the direction in which the tennis ball is moving as it hits the ground.

(1)

(c) Hence find the acute angle at which the tennis ball hits the ground, giving your answer in degrees to one decimal place.

(4)

The net of the tennis court lies in the plane $\mathbf{r}.\mathbf{j} = 0$

(d) Find the position of the tennis ball at the point where it is in the same plane as the net.

(3)

The maximum height above the court of the top of the net is 0.9 m.

Modelling the top of the net as a horizontal straight line,

(e) state whether the tennis ball will pass over the net according to the model, giving a reason for your answer.

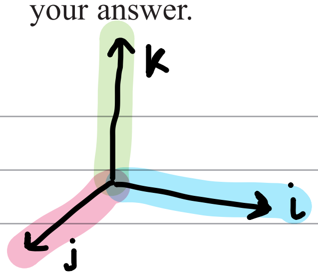
(1)

With reference to the model,

(f) decide whether the tennis ball will actually pass over the net, giving a reason for your answer.

(2)

a)



$$r = \begin{pmatrix} -4.1 + 9\lambda - 2.3\lambda^2 \\ -10.25 + 15\lambda \\ 0.84 + 0.8\lambda - \lambda^2 \end{pmatrix}$$

P 6 8 7 3 0 A 0 2 2 3 2

Question 6 continued

For the ball to hit the ground, the vector in the vertical direction (k) must be 0

$\therefore 0.84 + 0.8\lambda - \lambda^2 = 0$

$\lambda = \frac{7}{5} \cup \cancel{-\frac{3}{5}}$ ← reject negative solution because $\lambda \geq 0$

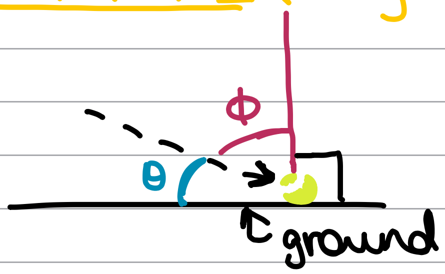
b) When hits the ground, $\lambda = 1.4$

Direction of travel : $(9 - 4.6(1.4))i + 15j + (0.8 - 2(1.4))k$

$= 2.56i + 15j - 2k$

METHOD 1 (using dot product):

c)



θ is angle at which ball hits the ground $= 90 - \phi$

Direction vector of perpendicular to the ground is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{\cos\phi = \frac{a \cdot b}{|a||b|}}$$

$$\cos\phi = \frac{\begin{pmatrix} 2.56 \\ 15 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{2.56^2 + 15^2 + (-2)^2}\sqrt{0^2 + 0^2 + 1^2}} = \frac{-2}{\sqrt{235.5536}} = -0.1303$$

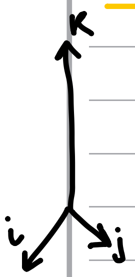
$\phi = \cos^{-1}(-0.1303) = 97.5°$

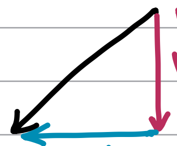
$\theta = 90 - \phi = 90° - 97.5° = -7.5°$

$\therefore$ hits the ground at $7.5°$

Question 6 continued

METHOD 2 (using trigonometry)

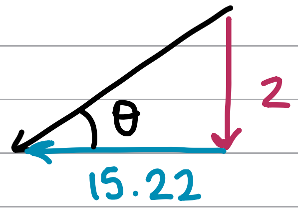




magnitude of vector in the vertical direction = 2

vector in horizontal plane is combination of i & j

$\sqrt{2.56^2 + 15^2} = 15.22$



$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{2}{15.22} = 0.131$

$\tan^{-1}(0.131) = 7.5°$

d) when the position of the tennis ball is in the same plane as the net

$\text{position} \cdot j = \text{position} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

$\begin{pmatrix} -4.1 + 9\lambda - 2.3\lambda^2 \\ -10.25 + 15\lambda \\ 0.84 + 0.8\lambda - \lambda^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -10.25 + 15\lambda = 0$

$\lambda = \frac{41}{60}$

$\therefore$ position : $\begin{pmatrix} -4.1 + 9\left(\frac{41}{60}\right) - 2.3\left(\frac{41}{60}\right)^2 \\ -10.25 + 15\left(\frac{41}{60}\right) \\ 0.84 + 0.8\left(\frac{41}{60}\right) - \left(\frac{41}{60}\right)^2 \end{pmatrix}$

$= \begin{pmatrix} 0.976 \\ 0 \\ 0.920 \end{pmatrix} = 0.976i + 0.920k$

P 6 8 7 3 0 A 0 2 4 3 2

Question 6 continued

e) When the ball is in the plane of the net, its height is 0.920 m above the ground (from (d))

$0.92 > 0.9 \therefore$ yes, ball passes over net

f) Because the ball is not a particle, but does have dimensions, and the ball's height is only 2 cm above the net, this is likely to be smaller than the balls diameter.
└ This means the ball will probably clip the net, but may still cross over

(Total for Question 6 is 13 marks)

INDUCTION: – Prove true for base case
– Assume true for $n=k$
– consider $n=k+1$ & replace by assumption
– Conclusion

7. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^n = \begin{pmatrix} 1-6n & 9n \\ -4n & 1+6n \end{pmatrix}$$

(6)

BASE CASE: For $n = 1$

$$\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}^1 = \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1-6(1) & 9(1) \\ -4(1) & 1+6(1) \end{bmatrix} = \begin{bmatrix} 1-6 & 9 \\ -4 & 1+6 \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$$

LHS = RHS

$\therefore$ statement is true for $n=1$

Assume statement true for $n=k$

assume $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}^k = \begin{bmatrix} 1-6k & 9k \\ -4k & 1+6k \end{bmatrix}$

$$\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}^{k+1} = \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}^k \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$$ ← replace by assumption

$$= \begin{bmatrix} 1-6k & 9k \\ -4k & 1+6k \end{bmatrix} \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5(1-6k) - 4(9k) & 9(1-6k) + 7(9k) \\ -5(-4k) - 4(1+6k) & 9(-4k) + 7(1+6k) \end{bmatrix}$$

$$= \begin{bmatrix} -5-6k & 9+9k \\ -4-4k & 7+6k \end{bmatrix}$$

$$= \begin{bmatrix} 1-6(k+1) & 9(k+1) \\ -4(k+1) & 1+6(k+1) \end{bmatrix}$$

Hence, statement is true for $n=k+1$

Since statement is true for $n=1$, and we have proved if true for $n=k$, it is true for $n=k+1$, thus by mathematical induction, the result holds true for all $n \in \mathbb{N}$

8.

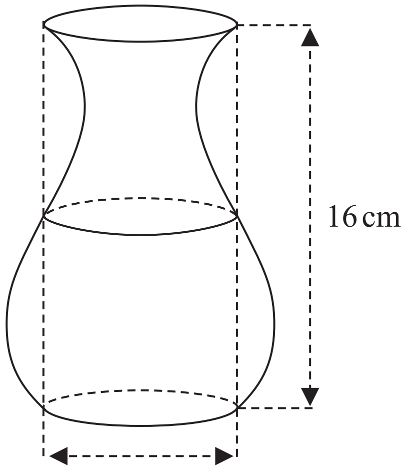


Figure 1

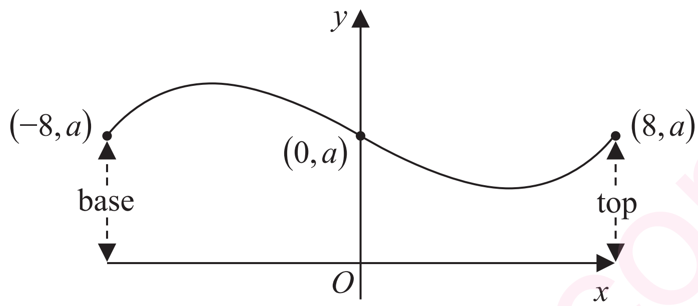


Figure 2

Figure 1 shows a sketch of a 16 cm tall vase which has a flat circular base with diameter 8 cm and a circular opening of diameter 8 cm at the top.

A student measures the circular cross-section halfway up the vase to be 8 cm in diameter.

The student models the shape of the vase by rotating a curve, shown in Figure 2, through 360° about the x-axis.

(a) State the value of a that should be used when setting up the model. **(1)**

Two possible equations are suggested for the curve in the model.

$$\text{Model A} \qquad y = a - 2\sin\left(\frac{45}{2}x\right)^{\circ}$$

$$\text{Model B} \qquad y = a + \frac{x(x-8)(x+8)}{100}$$

For each model,

(b) (i) find the distance from the base at which the widest part of the vase occurs,

(ii) find the diameter of the vase at this widest point. **(7)**

The widest part of the vase has diameter 12 cm and is just over 3 cm from the base.

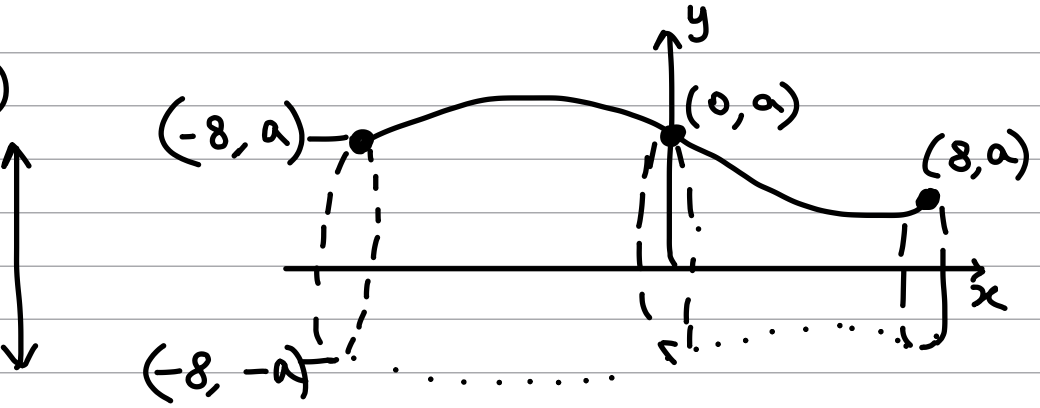
(c) Using this information and making your reasoning clear, suggest which model is more appropriate. **(1)**

(d) Using algebraic integration, find the volume for the vase predicted by Model B. You must make your method clear. **(5)**

The student pours water from a full one litre jug into the vase and finds that there is 100 ml left in the jug when the vase is full.

(e) Comment on the suitability of Model B in light of this information. **(1)**

a)



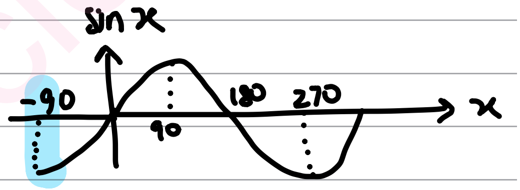
For the diameter of the base to be 8

$\therefore$ a must be 4

$a = 4$

b) For MODEL A:

(i) $y = a - 2\sin\left(\frac{45}{2}x\right)^\circ$



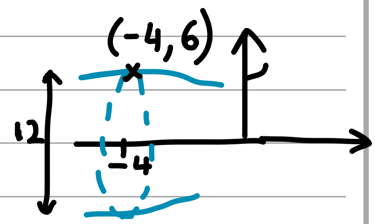
max y when $\sin\left(\frac{45}{2}x\right)^\circ = -1$

$\therefore \frac{45}{2}x = -90$

$x = -4$

$\therefore$ widest point is 4 cm in front of the base

(ii) Width at $x = -4$:



$y = 4 - 2\sin(-90)^\circ = 6$ cm

$\therefore$ width $= 2 \times 6 = 12$ cm

For MODEL B:

(i) $y = 4 + \frac{x^3 - 64x}{100}$

To find max y, differentiate and equate to 0

$\frac{dy}{dx} = \frac{3x^2 - 64}{100} = 0$

$x = \pm\frac{8\sqrt{3}}{3} \quad (\approx \pm 4.62)$

Distance from base $(x = -8)$, distance $= -8 - \left(\frac{-8\sqrt{3}}{3}\right)$

$= 3.38$ cm

(ii) $y = 4 + \frac{(-4.62)^3 - 64(-4.62)}{100}$

$= 5.97$ cm

$\therefore$ diameter $= 2 \times 5.97 = 11.9$ cm

c) Although both diameters are almost 12 cm,

Model B's distance to the base is closer to the specified distance of 3, so is more appropriate

d) $V = \pi \int_a^b y^2 \, dx$

$V_B = \pi \int_{-8}^{8} \left(4 + \frac{x^3 - 64x}{100}\right)^2 dx$

$= \frac{\pi}{10000} \int_{-8}^{8} (400 + x^3 - 64x)^2 \, dx$

$= \frac{\pi}{10000} \int_{-8}^{8} (160000 + 400x^3 - 25600x + 400x^3 + x^6 - 64x^4 - 25600x - 64x^4 + 4096x^2) \, dx$

Question 8 continued

$$= \frac{\pi}{10000} \int_{-8}^{8} (x^6 - 128x^4 + 800x^3 + 4096x^2 - 51200x + 160000)\,dx$$

$$= \frac{\pi}{1000} \left[\frac{x^7}{7} - \frac{128x^5}{5} + 200x^4 + \frac{4096}{3}x^3 - \frac{25600x^2}{2} + 160000x \right]_{-8}^{8}$$

$$= \frac{\pi}{10000} \left(1439783 - (-1439783) \right)$$

$$= 905 \text{ cm}^3$$

e) $905 + 100 = 1005$ ml

$\therefore$ roughly 1 L, so model is suitable